A Mathematical Framework to Assess Vulnerabilities in Co-Dependent Infrastructure and Natural System Networks

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ABSTRACT
The objective of this paper is to understand how flow and topology affect resilience and robustness of co-dependent infrastructure and natural system networks. This paper represents these co-dependent infrastructure and natural systems as an integrated network system (INS) and develops metrics to measure resilience and robustness based on the understanding of network connections and flow characteristics. The paper presents a mathematical framework founded in system dynamics and viability theory that can be used to formulate metrics of robustness and resilience for an INS. As each of these factors are a function of the system “structure”, a graph theoretic approach is used to represent and analyze the INS. The research builds on recent work indicating that the topology of a network determines its vulnerabilities. The immediate contribution of this paper is that it lays the mathematical foundations for measuring robustness and resilience of the INS, and identifies methods to characterize its topology and flow through it. In the long run, this research is likely to support a platform that will pro-actively identify and prepare for unexpected disruptions to infrastructure services while reducing undesirable environmental hazards.

INTRODUCTION
Engineered systems are designed to operate within a region of stability. Outside forces can push a system (network) outside its range of tolerance and cause the system to become unstable and potentially fail or offer a reduced level of service. Of the many possible forces that can affect infrastructure systems such as water pipe networks or power grid networks, the drive to interconnect these systems has led to an emergence of new potential forces that can also push these network systems outside of their stability envelope. Such dependencies can be in the form of control signals such as electric power grids requiring internet communications for Supervisory Control and Data Acquisition (SCADA) or dependencies of proximity as when a water main break disrupts a transportation network.

In many cases, these networks were designed to operate independently in the past, but have over time - for reasons of convenience or through deliberate design - become physically attached to other networks. As a result many different critical nodes of co-dependence have emerged. Often these nodes are apparent - for example a major multi-modal transportation hub and their management is well understood. Of
concern are the codependent nodes that are not deliberately planned, but are an outcome of multiple interacting independent decisions. Such nodes make the system vulnerable to catastrophic failure across all or some sub-systems.

For example, in August 2007 when the eight-lane I-35W bridge over the Mississippi river in Minneapolis collapsed, traffic crossing over the bridge, water traffic on the river, and power system connections that ran under the bridge were all disrupted. Infrastructure networks are also situated within the natural environment, and often interact with natural resource flows. Many of these co-dependencies emerge when taking advantage of existing permits, right-of-way etc. As a result, the vulnerabilities they create are seldom discovered until a catastrophic event occurs. These emergent vulnerabilities have increased the need to understand how networks are designed and how their stability changes as they expand over time, and how cross network effects from new inter-network connections affect network stability.

It is also important to recognize that codependent infrastructure networks are situated within environmental contexts that are directly impacted by their fate. Indeed, with a growing emphasis on sustainability, a study of failure trends in infrastructure cannot ignore their environmental performance. Therefore this study intends to investigate the fate of the integrated network systems (INS); consisting of infrastructure and natural networks. Characterizing environmental systems as networks and analyzing them in combination with infrastructure networks allows a holistic assessment of interactions that directly impact the environment. For example, quality of the water run-off from a highway network directly impacts the regional watershed networks. Conversely, increased storm flows within a watershed network can disrupt service on a highway network. Investigating the INS provides an opportunity for a holistic assessment of the entire system.

The objective of this research is to investigate the claim that the sustainability of such an integrated network can be defined as a function of system robustness and resilience, subject to environmental and socioeconomic constraints. Specifically, this paper presents a mathematical framework that can be used to formally measure the system robustness and resilience as the state of the system’s dynamic stability and viability, and as a function of its network topology and flow dynamics. Within the context of this study, robustness is defined as the extent of a typical stress a system can tolerate before it fails, and resilience, as the time it takes for a system to adapt and recover post failure. Systems are dynamic and adaptive and move from one dynamic equilibrium to the next. Resilience is considered to be the time taken by a system to return to the same or a different stable equilibrium. The framework is theoretically founded on principles of System Dynamics and Viability Theory.

LITERATURE REVIEW

This section discusses research that investigates robustness and resilience of infrastructure systems - particularly efforts that explicitly consider the influence of network topology and flow. Investigations into the topology of infrastructure networks - such as telecommunication networks - show that physically they do not exhibit properties of random networks because they often tend to be deliberately designed (Dekker and Colbert 2004). However, in a recent study of the Italian power infrastructure, Buldyrev et al. (2010), suggested that the failure of a single power
station could cause a catastrophic failure in the entire network. Conceptually, given the nature of human settlements (and supporting infrastructure) and natural systems to self-organize and grow through preferential attachment, there is reason to believe that the integrated network may exhibit scale-free properties. Scale-free networks are more vulnerable to targeted attacks as their degree distribution is power law distributed, resulting in a small number of nodes that are highly connected. However, they fair better with random failures. Non-scale-free networks on the other hand are more vulnerable to random attacks than to targeted attacks. Barabasi et al. (1999) conjecture that all large networks self-organize into scale-free networks at some level, once the networks reach a certain size. However this claim is yet to be thoroughly investigated.

When it comes to assessing questions of flow, a model based on Leontief’s input/output based approach has been used to identify interdependencies and interconnectedness in between, and among different infrastructure systems (Haimes and Jiang 2001). It is important to note that while the Leontief matrix explicitly represents sector wide resource interactions, it does not relate the resource flows along any physical or natural infrastructure networks. With respect to flow through networks, the idea of changes in flow - beyond the capacity of the network - resulting in cascading system failures has been explored in the context of overloaded electrical power grids (Crucitti et al. 2004). There is recognition of a need to develop modeling approaches that account for flow through networks in addition to topology. However, the outcome has been limited to the explicit consideration of dynamics of flow through networks (Wang et al. 2005). There is also reason to believe that if indeed the topology of the integrated network has scale-free properties, then the load distribution across the network will also be power distributed (Goh 2001).

A third section of the literature explicitly models questions of vulnerability and cascading failure in particular or a combination of critical infrastructure networks such as power grids and telecommunication networks (Buldyrev et al. 2010; Ulieru 2007). Notably, Turner et al. (2003) propose a framework for vulnerability analysis that emphasizes not only the exposure of a network to hazard, but also its sensitivity and resilience to perturbations (Turner et al. 2003). Using a simulation based framework they applied their method to assess the sensitivity of a power transmission system to multiple hazards. While they do provide an integrated assessment of multiple dimensions in their measure of resilience, they do not explicitly account for environmental impacts. In addition, their effort does not explicitly account for interdependencies between multiple infrastructure and natural networks (Forrester 1991).

Among other notable efforts, Jordan and Javernick-Will, (2013) note that many disasters are the result of human and infrastructure dependencies. Therefore, simply restoring a system to its pre-disaster condition does not remove the vulnerabilities (the interactions), but leaves the system prone to future disasters. This directly hints to the need to study the underlying network topology and identify co-dependencies that are at the root of vulnerabilities. Jordan and Javernick-Will, used a Delphi study to determine the indicators of disaster recovery from journal papers that used the key words of “disaster recover, resilience, or vulnerability”. The results indicated that one of the most common and most important recovery metrics was
infrastructure status, as the infrastructure recovery is closely tied to both economic and social network recovery. While their research provides a good starting point it does not provide any quantitative system level metrics to measure robustness and resilience. This discussion establishes the following gaps in the literature:

- There are very few approaches that account for the network topology and flow dynamics when modeling interdependencies between infrastructure systems.
- While there are quite a few research efforts that address the vulnerabilities that arise due to interdependencies between coupled infrastructure networks, there are very few efforts that assess the coupling between infrastructure and natural networks.
- Finally, the literature points at the need to include considerations of resilience, robustness and vulnerability in assessing system sustainability, yet most of the studies fall shy of explicitly considering environmental impacts and outcomes.

The proposed research will use each of the above as points of departure using a formulation based in System Dynamics and Viability Theory to model a trade-off based decision-making framework.

THEORETICAL FRAMEWORK

The framework presented in this section is motivated by the need to develop a mathematical model for expressing system robustness and resilience as a function of a system’s dynamic stability and viability. By definition, a system illustrates dynamic stability if the dynamic response for a bounded input, is also bounded. This definition is very useful for measuring system robustness as a disruption in a system can be formalized as the unbounded response of a system to an input signal. Similarly, resilience can be measured as the time taken by the system to return to a state where the response is within a defined bound. In order to define the bounds of the system, the concept of viability must be introduced, i.e., the response of a system can be considered to be bounded as long as it is still viable. Therefore, system viability is defined as a family of states in which the system does not violate a set of given constraints.

Hence, there are two fundamental concepts to this theoretical formulation: (i) dynamic stability and (ii) dynamic viability of a system. A system illustrates dynamic viability if it continues to satisfy a set of constraints as time passes. Depending on their structure, some systems tend to move towards stable equilibrium points, which may not always satisfy all system constraints. This leads to the following notions: (i) a system is sustainable if it is stable with respect to a desired state, and viable given a set of constraints; and (ii) a sustainable policy is a dynamic control mechanism that minimizes the difference between the system state and the desired state, while ensuring that system constraints are not violated.

The next component of this formulation is the measure of flow through the system. One or more flows of interest can be modeled using different currencies. The dynamic viability and stability of the system will be a direct function of the maximum allowable flow through it. Hence, we introduce the notion of a maximum carrying capacity of a network for a given currency. This capacity can be set by a
physical constraint, for example, the traffic volume a highway network can support; or the constraint can be environmental. In each of these cases, a violation of the constraint will lead to an unbounded response (e.g., a congested network and a polluted watershed), making the respective systems unviable and therefore unsustainable.

In order to model the above notions, we use the population dynamics problem as a starting point. Population growth within this system is limited by available natural resources, and the carrying capacity metric (CCM), $C$, in such a system is the maximum population that a limited natural resources can sustain. This is strictly speaking a two-node one-edge network, and the currency flowing through the system is the number of people. Each person enters the system when they are born (first node), and leave the system when they die (second node). The time they are resident in the system is the time it takes them to flow through the edge from the first to the second node. The CCM is a measure of how many people can “flow” though the edge. The time rate change of population is defined by two sets of competing dynamic feedbacks: (i) If the fractional birth and death rates remain the same, as population increases, the number of births and deaths also increase, thus reinforcing or balancing the population stock respectively; (ii) The propensity of a prospering population to grow and reproduce keeps reinforcing the population till the CCM is exceeded, at which point the unavailability of resources forces the fractional death rate to increase thus balancing the population. These two competing feedbacks, as monitored by the ratio of population $P$ to carrying capacity ($P/C$), ensure that as time approaches infinity, the population asymptotically approaches the carrying capacity of the system - a stable equilibrium - after exhibiting S-shaped growth over time. This is mathematically expressed by the logistic model of growth, as follows:

$$\frac{\delta P(t)}{\delta t} = gP[1 - \frac{P}{C}]$$

A point of interest in the above system is the point of inflection (often referred to as the tipping point) at $P = C/2$, where the balancing feedback becomes equal to, the reinforcing feedback, implying that the rate of change of the state of the system with respect to itself is 0. Beyond this point the balancing loop feedback starts dominating - leading to the eventual equilibrium condition. This mathematical equilibrium, though stable, is not a sustainable system state as the population exhaustively uses available resources to survive. Instead, the point before $P = C/2$, is attractive as a point of sustainability as it sustains growth without reaching the resource limit. This is indeed a mathematical rendition of the definition of sustainability in the Brundtland Commission’s report (1987), and echoed in more recent work (Mihelcic et al. 2003).

Next, Viability Theory concepts (Aubin 2006) are used to support this formulation as it accounts for complex system organization, and allows development of control strategies that maintain system stability and viability. Mathematically speaking, it defines a viable sub-space within which the system can exist in a stable fashion without violating any of the system constraints. Hence, it is useful for
representing context specific resource constraints (Eisnack et al. 2007; Aubin 2006). It is complemented by the ability of system dynamics methods to represent reciprocal causal relationships between socioeconomic activity and use of resources. A marriage of Viability Theory to this System Dynamics problem allows explicit modeling of resource constraints. They can represent system structure and relationships, and afford investigation of dynamic behavior. This formulation models the above notions, and allows further investigation of decision-making as a dynamic control problem.

Hence, the formulation is extended to include an upper bound on available natural resources $N$, and a per capita consumption, $R$, allowing the expression of the viability constraint $P \times R \leq N$. It forces the net system consumption to be less than the available resources. It gives policy makers two control variables, $P$ and $R$, and the ability to make trade-offs between them. They can reduce consumption to accommodate a higher population, or control population to sustain higher consumption.

The formulation can be expressed as: Given a set of control variables $C=\{P, R\}$, and system viability criteria $V=\{P \times R \leq N\}$. The system is said to be sustainable if it is stable and viable. The system is defined as stable if the population, response $P$ is in the interval $[C_1, C_2]$, where $C_1 \leq C/2 \leq C_2$, and per capita consumption response $R$ is in the interval $[R_1, R_2]$. This allows the decision-maker the ability to trade-off between the control variables $P$ and $R$. Further, the assumption of finite resources can be relaxed if technological innovation allows an increase in the pool of available natural resources, setting the viability criteria to $P \times R \leq (N + \Delta N)$ and introducing a third control. The CCM when viable for a given viability criteria, ensures that critical system constraints will be sustained. Therefore the research challenge is to use this expressive control theoretic formulation to compare the dynamics of alternative control measures and hence the sustainability of alternative policies.

Networks will be represented using a simulation platform developed at Montana State called Network Exchange Objects (NEO). The NEO framework allows multiple networks with varying types of flows to be represented together in different layers and be interconnected. It can perform simulations to determine how restrictions on one network can affect the flows on adjacent networks. Further, transient behavior can be observed by causing currency impulses at various points in the system to help characterize the impulse response of the system. NEO can simulate multiple currency flows and the network interactions between one or more infrastructure and natural networks.

APPLICATION OF NEO PLATFORM

The goal of this section is to introduce the NEO platform and establish its suitability for implementing the above theoretical framework. Some simple experimental models were tested in NEO, which uses nodes and edges as the fundamental building blocks. Nodes represent control points in the system through which flow is changed or directed, e.g., junctions in a transportation system such as a road intersection. The nodes have properties defining their type, the inflow and outflow of currency, and any storage capacity. Edges represent the links between these intersections (roads, for example). The edges are defined by their starting and
ending nodes, and by the functions performed on the currencies between nodes - the equations of flow for example in water pipes. Currencies are the objects of flow (water, traffic, electric power) through the networks. Simple flow equations were assigned to the edges to regulate the speed of the currency transfer based on the differential concentrations of currencies between the connected nodes. Each node could be monitored at each time step allowing for a detailed step-by-step analysis of the state of the network, if desired. NEO has been used previously to model stream flows and microbial carbon uptake where water flow, dissolved organic carbon form two dependent networks (Helton et al. 2010).

Just as nodes and edges are the building blocks of NEO, simple network topology and flow are used to construct complex networks. This paper began constructing simple networks to understand their operation, which can later be combined to more complex systems.

The following topologies and flow patterns were investigated:

- The first model was a simple linear network with 4 nodes and 3 edges of the same type representing 3 pipe segments and 4 intersections as shown in Figure 1. Two flow patterns were setup within it: (i) flow currencies in the simulation were evenly distributed between all of the nodes, and (ii) the currencies diffused through the network and collected in the end (downstream) node. Currencies were first only placed in the first cell in the network and allowed to diffuse throughout the network. Next, currencies were initially placed in two cells in the network and allowed to diffuse.

- The next model, as shown in Figure 2, was created with one central node and 4 identical nodes connected to the central node. When simulated, this network evenly distributed the currencies from the central node to the other 4 connected outer nodes. In addition, in this model varying flow conditions were explored: (i) declaration of initial conditions: each edge having different levels of currencies at the beginning of the simulation (time t = 0), and (ii) declaration of flow probabilities: the currencies distributed themselves out of a node according to a predefined set of probabilities for each outgoing edge.

- A third model was created using two parallel branches emanating for a common starting node (Figure 3). When the probabilities of currency
entering each branch were the same \((p = 0.5)\), the currencies flowed between the two branches and collected in the common end node. Models like this can be used to emulate a redundant pipe network that is connected at both ends where the water had an equal chance of flowing through either branch.

Figure 4 provides a description of how currencies diffuse through the network when originating at a single point in the network or from more than one point in the network respectively. As the plots suggest, the behavior of flow in the network is a function of the edge behaviors and also node characteristics. If the carrying capacity of the network is adjusted, the rate of diffusion reflects this new restriction. A linear network (similar to figure 1) was constructed with the connecting edges adjusted to have differing carrying capacities (infinite, 4, 2, and 1). The simulation illustrates how the carrying capacity can affect the flow rate through the network. Figures 5 and 6 show the results.

While this is a preliminary illustration of how NEO can be used within the proposed theoretical framework, current and future work is addressing ways of
modeling complex overlapping networks with multiple interactions and co-dependencies as illustrated in Figure 7. Such a network can be used to represent a system where each network has 4 primary nodes and also interacts with the adjacent networks just as infrastructure networks can interact with each other.

![Figure 7. Example of network with dependencies](image)

**CONCLUSION**

The modeling method described was successful in using different network topologies with the same node and edge types. These provide an appropriate base for starting to model systems with multiple interactions and constraints. Ultimately, the research is using GIS data for existing networks. This paper outlines only the first steps taken in the process of determining vulnerabilities in dependent infrastructure system networks. Modeling these networks with the future steps indicated and then simulating the currency flows will permit the researchers to then create failures in the networks and to examine performance using metrics of resilience and robustness. This research will contribute to the field, information for planners managing their infrastructure and disasters response agencies as they can identify areas of vulnerability in their interconnected networks. This interconnection can disrupt the flow on other networks, compounding the incident severity.

**REFERENCES**


